Postbuckling Behavior of Orthotropic Cylinders under Axial Compression

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Solutions of the classical buckling problem for orthotropic cylindrical shells have long been available, but the practical applicability of such solutions is questionable. Results of a finite-displacement analysis of axially compressed orthotropic cylinders are presented here. These results serve to indicate the range of applicability of classical-theory analyses. It is shown that for certain parameter combinations the minimum postbuckling equilibrium load is reasonably close to the classical buckling load and that the two loads may coincide. In such cases, the classical theory is applicable.

Nomenclature

a_i = coefficients in displacement function, Eq. (1)
$A_{11}, A_{12}, A_{22}, A_{33}$ = see Eq. (2)
$D_{11}, D_{12}, D_{22}, D_{33}$ = see Eq. (2)
E = Young's modulus
F = stress function
G = shear modulus
L = length of shell
l_x, l_y = axial and circumferential half-wavelengths
M_x, M_y, M_{xy} = bending and torsional moments per unit width
m = L/l_y
N_x, N_y, N_{xy} = axial, circumferential, shear force per unit width at middle surface of shell
N = applied compressive load per unit width
$\bar{N}$ = load parameter, Eq. (3)
$\bar{N}_{cr}$ = critical value of $\bar{N}$
N_{cl} = critical value of N according to classical theory
N_{min} = minimum value of N in postbuckling range
r = radius of cylinder
t = shell thickness, monocoque shell
u, w = axial and radial displacement, w is positive inward
V = total potential energy
x, y = axial and circumferential coordinates on middle surface of shell
x_i = displacement parameters, Eq. (11)
α = shell parameter, Eq. (4)
β = wavelength parameter, Eq. (8)
γ = shell parameter, Eq. (3)
γ_{xy} = shear strain at middle surface of shell
$\epsilon_x \epsilon_y$ = axial and circumferential strains at middle surface of shell
η = wavelength parameter, Eq. (8)
$\eta_P \eta_S$ = shell parameters, Eq. (3)
λ_{ij} = coefficients, Eq. (10)
ν = Poisson's ratio
x, y = as subscripts, after a comma, indicate partial differentiation of the principal symbol with respect to x or y

Introduction

The immense size of launch vehicles now under development will require use of cylindrical shells of sandwich or stiffened construction. Unfortunately, reliable methods to determine the critical axial load for such shells are not available. Solutions based on the classical theory have been derived,[1-3] but the applicability of this theory is questionable. Some tests have given results that appear to be in agreement with the theory, but other tests have shown a marked disagreement, and there seems to be no information in the literature on the limits of applicability of the classical theory.

It is generally assumed that the discrepancy between tests and theory for the axially compressed cylindrical shell is due to the character of its postbuckling behavior. Therefore, a finite displacement analysis of orthotropic cylinders is likely to throw some light on the question of the range of applicability of the classical theory. Results of such an analysis are presented here, and the adequacy of the classical theory is discussed in view of these results.

The theory presented here is derived for cylinders of orthotropic material. The material properties may vary with the normal coordinate, but not with the surface coordinates. The theory may be applied, of course, to stiffened shells provided the stiffeners are sufficiently close together, and to sandwich shells if the core is sufficiently stiff. The results of Ref. 1 indicate that the effect of stiffener eccentricity with respect to the skin's middle surface may be significant. However, for simplicity, this effect is omitted here.

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Analysis

An analysis of the postbuckling behavior of infinitely long orthotropic cylinders under axial compression was presented in Ref. 3. However, that analysis is somewhat inaccurate. It was shown in Ref. 4 that the assumed form of the buckling pattern contains too few terms, so that for the special case of a monocoque shell it gives a minimum equilibrium load in the postbuckling range which is about three times too high. A Rayleigh-Ritz-type analysis will be utilized here in a more accurate study of the postbuckling behavior. Of the different displacement functions used in Ref. 4, the most accurate contains 9 terms and gives a minimum postbuckling load equal to 0.11 times the classical buckling load. This result is in close agreement with experimental evidence, but, for economy in the numerical analysis, some accuracy will be sacrificed here by use of the 5-term displacement function:

$$w = a_0 + a_1 \cos\frac{2\pi x}{l_x} + a_2 \cos\frac{\pi x}{l_x}\cos\frac{\pi y}{l_y} + a_3 \cos\frac{4\pi x}{l_x} + a_4 \cos\frac{2\pi x}{l_x}\cos\frac{2\pi y}{l_y} + a_5 \cos\frac{3\pi x}{l_x}\cos\frac{3\pi y}{l_y} \quad (1)$$

For the monocoque cylinder, this function yields a ratio between the minimum postbuckling load and the classical buckling load of 0.12.

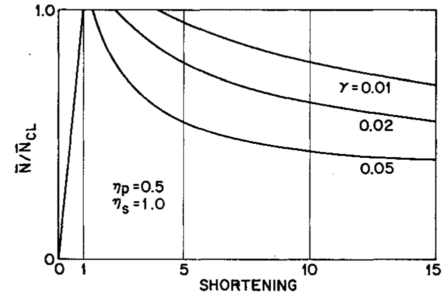


Fig. 1 Load-displacement curves, ring-stiffened cylinders.

The orthotropic behavior of the shell wall is governed by the elastic constants defined in the following equations:

$$\left.\begin{aligned} \epsilon_x &= A_{11}N_x + A_{12}N_y \\ \epsilon_y &= A_{12}N_x + A_{22}N_y \\ \gamma_{xy} &= A_{33}N_{xy} \\ M_x &= D_{11}w_{,xx} + D_{12}w_{,yy} \\ M_y &= D_{12}w_{,xx} + D_{22}w_{,yy} \\ M_{xy} &= 2D_{33}w_{,xy} \end{aligned}\right\} \quad (2)$$

Here the A_{ij} are the inverse of the extensional stiffnesses, whereas the D_{ij} represent flexural stiffness. For the monocoque shell of isotropic material,

$$A_{11} = A_{22} = 1/Et \qquad A_{12} = -\nu/Et \qquad A_{33} = 1/Gt$$

$$D_{11} = D_{22} = Et^3/[12(1-\nu^2)]$$

$$D_{12} = \nu D_{11} \qquad D_{33} = Gt^3/12$$

If stiffeners are eccentrically attached to the shell, there will also be coupling between the equations for strains and changes of curvature. This effect is omitted here.

It was shown in Ref. 3 that the number of independent variables in the analysis of the infinitely long cylinder can be reduced to three through the following substitutions:

$$\left.\begin{aligned} \bar{N} &= (rN/2)(A_{11}/D_{22})^{1/2} \\ \eta_S &= (A_{12} + \tfrac{1}{2}A_{33})/(A_{11}A_{22})^{1/2} \\ \eta_P &= (D_{12} + 2D_{33})/(D_{11}D_{22})^{1/2} \\ \gamma &= (D_{11}A_{11})/(D_{22}A_{22}) \end{aligned}\right\} \quad (3)$$

where N is the axial compressive force per unit width of the circumference.

For cylinders of finite length, the following additional independent parameter must be introduced:

$$\alpha = \frac{L^2}{2rm^2\pi^2 A_{22}(D_{22}/A_{11})^{1/2}} \quad (4)$$

where m is the number of half-waves in the axial direction.

The three nondimensional shell wall stiffness parameters η_S, η_P, and γ, are defined through Eqs. (2) and (3). For the monocoque shell of isotropic material, $\eta_S = \eta_P = \gamma = 1$. Of these parameters, γ is the most influential. It is evident that with circumferential stiffening $\gamma < 1$, and with axial stiffening $\gamma > 1$. The parameter α introduces the effect of shell length. For the monocoque shell of isotropic material, and with one half-wave in the axial direction,

$$\alpha = \frac{[3(1-\nu^2)]^{1/2}}{\pi^2}\frac{L^2}{rt}$$

The equations of equilibrium and compatibility for the orthotropic cylinder were derived in Ref. 3. The compatibility equation is

$$A_{22}F_{,xxxx} + 2[A_{12} + (A_{33}/2)]F_{,xxyy} + A_{11}F_{,yyyy} = -(1/r)w_{,xx} + (w_{,xy})^2 - w_{,xx}w_{,yy} \quad (5)$$

where the stress function F is defined such that

$$N_x = F_{,yy} \qquad N_y = F_{,xx} \qquad N_{xy} = -F_{,xy} \quad (6)$$

While equilibrium in the plane of the shell will be assured by the introduction of this stress function, the third equilibrium equation will not be used, but equilibrium configurations will be established by use of the principle of stationary potential energy.

For the total potential energy of the system, Ref. 3 gives

$$V = \frac{1}{2}\int_0^L\int_0^{2\pi r}\{A_{11}F_{,yy}{}^2 + 2A_{12}F_{,yy}F_{,xx} + A_{22}F_{,xx}{}^2 + A_{33}F_{,xy}{}^2\}dxdy + \frac{1}{2}\int_0^L\int_0^{2\pi r}\{D_{11}w_{,xx}{}^2 + 2D_{12}w_{,xx}w_{,yy} + D_{22}w_{,yy}{}^2 + 4D_{33}w_{,xy}{}^2\}dxdy - \int_0^{2\pi r}(N_x)_{x=L}dy\int_0^L u_{,x}dx \quad (7)$$

The parameters defined in Eq. (3) are introduced together with

$$\left.\begin{aligned} \eta &= (2r\pi^2/l_y{}^2)(A_{22}D_{22})^{1/2} \\ \beta &= (l_y/l_x)(A_{22}/A_{11})^{1/4} \end{aligned}\right\} \quad (8)$$

Substitution of the displacement function, Eq. (1), into Eq. (5) gives F in terms of the generalized coordinates of the problem, $a_i(1 \leq i \leq 5)$, l_x, and l_y. The coefficient a_0 in Eq. (1) can be determined from the relation between strains and displacements and the condition that the tangential displacement be continuous. With w as well as F expressed in

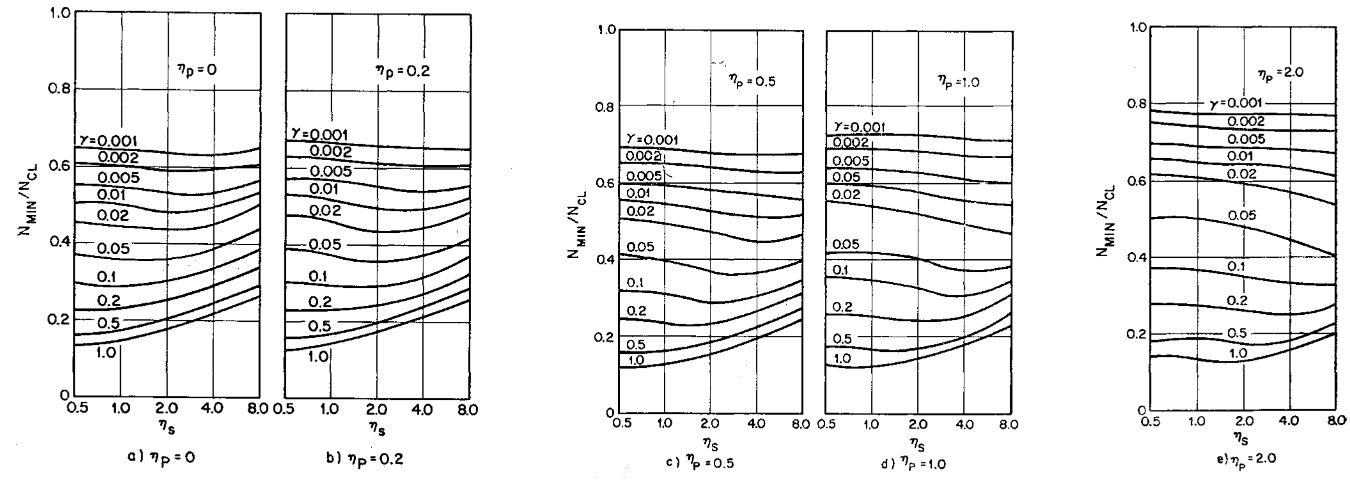
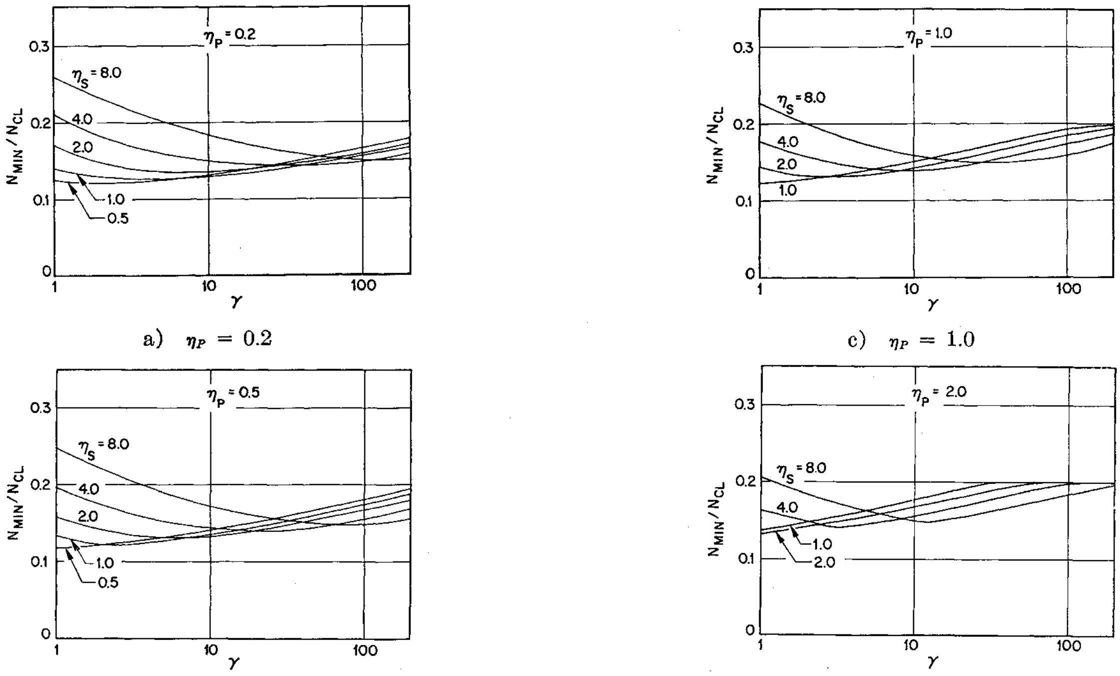


Fig. 2 Minimum postbuckling load, long cylinders ($\gamma \leq 1.0$).



a) $\eta_P = 0.2$ b) $\eta_P = 0.5$ c) $\eta_P = 1.0$ d) $\eta_P = 2.0$

Fig. 3 Minimum postbuckling load, long cylinders ($\gamma \geq 1.0$).

terms of the generalized coordinates, the potential energy of the system can be written

$$(4r/\pi LD_{22})V = 4(\beta/\eta)^2[\beta^2x_2^2(32\lambda_{20}x_1^2 + \lambda_{11} + 512\lambda_{40}x_3^2 + 16\lambda_{22}x_4^2 + 81\lambda_{33}x_5^2) - x_2^3\beta^2(8\lambda_{20} + x_1 + 4\lambda_{11}x_1 + 512\lambda_{40}x_3x_4^2 + 256\lambda_{22}x_3x_4^2) + x_2^4\beta^2(\tfrac{1}{2}\lambda_{20} + 4\lambda_{11}x_1^2 + 128\lambda_{02}x_1^2x_4^2 + 16\lambda_{02}x_1x_4 + \tfrac{1}{2}\lambda_{02} + 128\lambda_{40}x_4^4 + 4\lambda_{31}x_1^2 + 32\lambda_{31}x_1x_3 + 16\lambda_{31}x_1x_4 + 64\lambda_{31}x_3^2 + 64\lambda_{31}x_3x_4 + 16\lambda_{31}x_4^2 + 1024\lambda_{22}x_3^2x_4^2 + 324\lambda_{13}x_1^2x_5^2 + 144\lambda_{13}x_1x_4x_5 + 2592\lambda_{13}x_1x_3x_5^2 + 16\lambda_{13}x_4^2 + 576\lambda_{13}x_3x_4x_5 + 5184\lambda_{13}x_3^2x_5^2 + 128\lambda_{04}x_4^4 + \tfrac{6561}{2}\lambda_{60}x_5^4 + 64\lambda_{51}x_3^2 + 576\lambda_{51}x_3x_4x_5 + 1296\lambda_{51}x_4^2x_5^2 + 64\lambda_{42}x_1^2x_4^2 + 144\lambda_{42}x_1x_4x_5 + 81\lambda_{42}x_5^2 + 81\lambda_{24}x_5^2 + 1296\lambda_{15}x_4^2x_5^2 + \tfrac{6561}{2}\lambda_{06}x_5^4 + 1024\lambda_{62}x_3^2x_4^2 + 324\lambda_{53}x_1^2x_5^2 + 5184\lambda_{73}x_3^2x_5^2)] + x_2^2[\gamma\beta^4(32x_1^2 + 1 + 512x_3^2 + 16x_4^2 + 81x_5^2) \times (1 + 2\eta_p\beta^2\gamma^{1/2})(1 + 16x_4^2 + 81x_5^2)] - 8(\beta^2/\eta)x_2^2\bar{N}(4x_1^2 + \tfrac{1}{2} + 16x_3^2 + 2x_4^2 + \tfrac{9}{2}x_5^2) + \text{const} \quad (9)$$

where

$$\lambda_{ij} = 1/(j^4 + 2i^2j^2\eta_S\beta^2 + i^4\beta^4) \quad (10)$$

and

$$\left.\begin{aligned} x_1 &= a_1/a_2 \\ x_2 &= a_2r\pi^2/l_y^2 \\ x_3 &= a_3/a_2 \\ x_4 &= a_4/a_2 \\ x_5 &= a_5/a_2 \end{aligned}\right\} \quad (11)$$

If only terms of second order in x_i are retained in Eq. (9), and x_i is set equal to zero when $i \neq 2$, an expression for the potential energy is obtained from which the classical buckling load for infinitely long cylinders can be obtained. It is found after minimization with respect to the parameter η that

$$(\bar{N}_{cr})^2 = (1 + 2\eta_P\gamma^{1/2}\beta^2 + \gamma\beta^4)/(1 + 2\eta_S\beta^2 + \beta^4) \quad (12)$$

where the parameter β is chosen to minimize $\bar{N}_{cr}$. This equation is in agreement with the solution for the classical load in Ref. 3.

If $l_x = L/m$ is substituted into Eq. (9), a corresponding equation for the classical buckling load of finite length cylinders is obtained:

$$\bar{N}_{cr} = \alpha\beta^4/(1 + 2\eta_S\beta^2 + \beta^4) + (1 + 2\eta_P\beta^2 + \gamma\beta^4)/(4\alpha\beta^4) \quad (13)$$

where β is obtained from the equation

$$4\alpha^2\beta^8(1 + \eta_S\beta^2) = (1 + \eta_P\gamma^{1/2}\beta^2)(1 + 2\eta_S\beta^2 + \beta^4)^2 \quad (14)$$

and m is chosen to minimize $\bar{N}_{cr}$.

In a study of postbuckling behavior of the shell, all terms in Eq. (9) are retained. The derivatives of V with respect

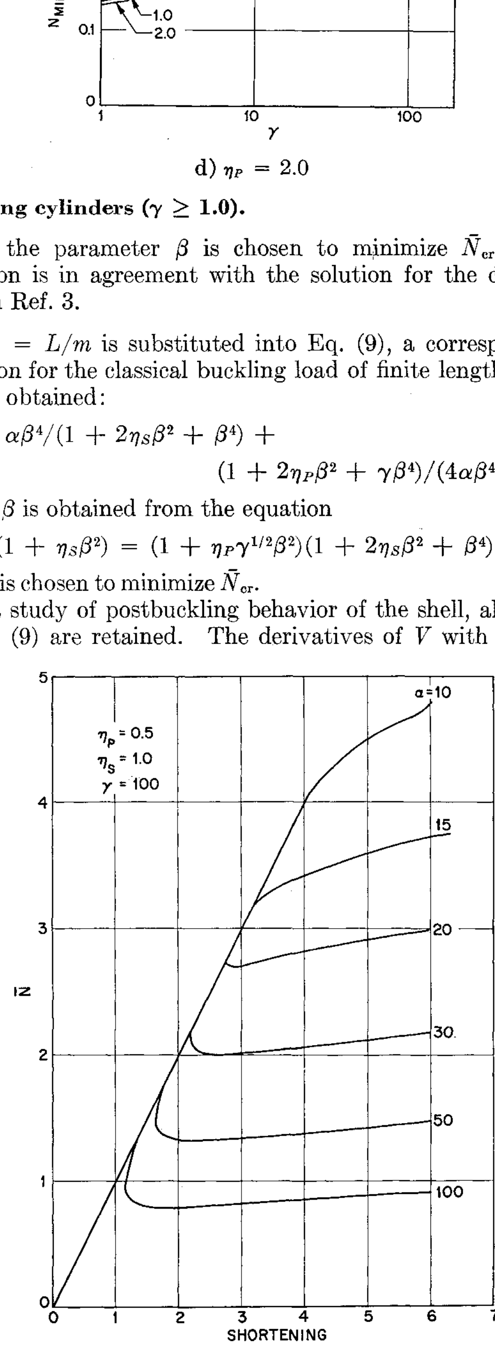


Fig. 4 Load displacement curves for short cylinders.

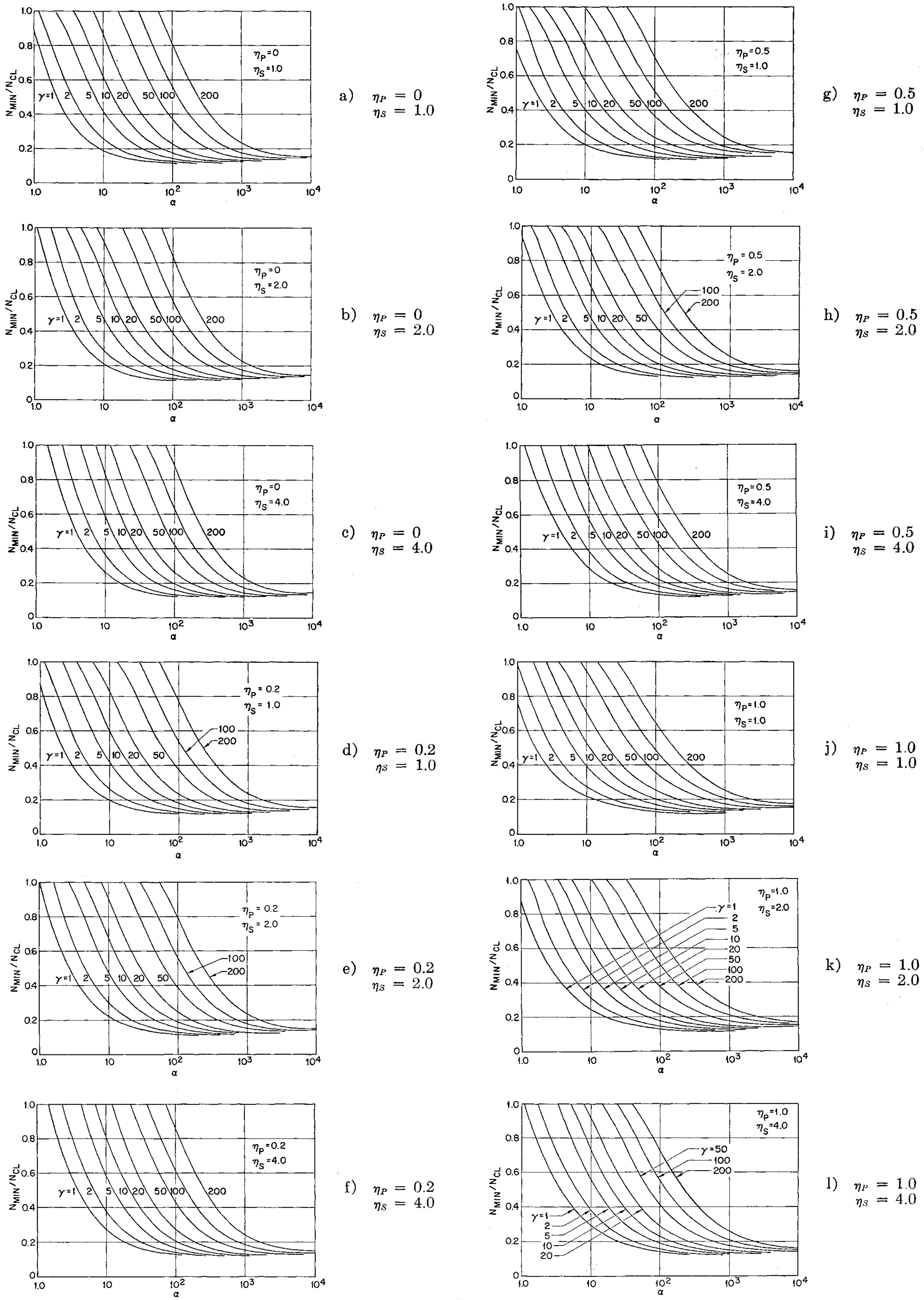


Fig. 5 Minimum postbuckling load, short cylinders.

to the generalized coordinates of the system yield, when set equal to zero, seven nonlinear algebraic equations. Simultaneous solutions to these equations define equilibrium configurations and can be found by use of the Newton-Raphson iteration method.[4] In the numerical analysis, the IBM 7090 was used, and the postbuckling behavior was studied within a wide range of the independent parameters. It was found that the only case of practical interest in which the postbuckling behavior appreciably deviates from that of the monocoque cylinder is the case for which $\gamma < 1$, i.e., the ring-stiffened cylinder. Some load-displacement curves for such cylinders are shown in Fig. 1, and the minimum postbuckling load for different parameter combinations can be obtained from Fig. 2. The curves are normalized with respect to the classical buckling load as defined by Eq. (12). For cylinders with $\gamma > 1$, i.e., stringer-stiffened cylinders, the minimum postbuckling load is shown in Fig. 3.

The preceding analysis indicates that, for cylinders with $\gamma \gg 1$ (stringer-stiffened cylinders), the axial wavelength of the diamond pattern is very large. In practical applications, this buckling pattern cannot be accommodated within the length of the shell. To determine accurately the postbuckling behavior of a cylinder with finite length appears to be a formidable task. However, a conservative estimate of the minimum postbuckling load can easily be obtained by modification of the analysis for infinitely long shells. If the axial half-wavelength is set equal to the length of the shell, the number of generalized coordinates is reduced to six, while the number of independent parameters is added to α as defined in Eq. (4).

The approximate equations thus obtained for the cylinder of finite length were also solved numerically by use of the IBM 7090. Some typical load displacement curves are shown in Fig. 4. The minimum postbuckling load can be found as a function of the shell parameters in Fig. 5. The N_{cl} is here based on Eqs. (13) and (14). The number of half-waves in the axial direction m should be chosen such that the minimum postbuckling load is minimized. For short cylinders, m will be equal to one. Whenever $m = 2$ gives a lower value for N_{min} than does $m = 1$, the analysis for infinitely long cylinders gives a sufficiently close estimate of this value.

Conclusions

The classical buckling load, of course, is an upper bound for the actual critical load of axially compressed cylindrical shells. On the other hand, the minimum postbuckling equilibrium load is the lowest load under which a buckled configuration can be maintained and may, as such, be considered a lower bound. In general, these two bounds are far apart, but the present analysis shows that, in certain cases, the two bounds approach one another. One of these cases is the ring-stiffened cylinder. For this type of cylinder, experiments show only moderate scatter in buckling loads.[5]

From Fig. 4, it may be seen also that the character of the postbuckling behavior changes with the length of the shell. With decreasing cylinder length, a point will be reached at which the load displacement curve resembles the corresponding curve for flat plates. The upper and lower bounds then coincide and the classical buckling load should be applicable. Some test results for relatively short stringer-stiffened cylinders are available[6] which seem to indicate that the classical buckling load is applicable. It should be noted, however, that comparison between theory and tests is obstructed by difficulties in determination of the amount of edge restraint in the tests.

With a lower bound available it is, of course, possible to use this bound as a design limit. Although in some cases this will lead to reasonable results, there will be other cases in which a less conservative and more realistic estimate of the critical load is needed. Unfortunately, presently available experimental data are too sparse for the development of a practical method of analysis for orthotropic or stiffened shells which is entirely satisfactory for all parameter combinations.

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